A nonparametric Item Response Theory

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Why non-parametric: the role of monotonic relationship

- Ability is a latent variable
- It is measured by the successful responses to questions that differ in difficulty
- Adding (subtracting) a question to the questionnaire is equivalent to applying a monotonic transformation to “ability”

Alternative uses of non parametric "Item Characteristic Curve"

- Whether the question (item) belongs to the subject matter of the exam?
- Its discriminatory power
- Whether the question is redundant?

Item Response Theory

- It imposes a given distribution of abilities among the examinees and a given structure of the relationship between the success of answering a question and ability:
- $$p(S \mid \theta) = \frac{1}{1 + e^{-a(\theta - d)}}$$

where: θ is ability, *d* is question difficulty level,
a is item discrimination level,

P(S|θ) is the proportion of correct answers to a specific question with given ability θ

Item Response Theory (*Cont.*)

- The maximum discrimination ability of the question is imposed to be at the median of the distribution.
- The discrimination of the question is at the median of the distribution of the response.
- ICC (Item Characteristic Curve) presents the regression curve of probability of success and “ability”

LMA (Line of Independence Minus Absolute Concentration) Curve

- Similar to ICC, except
- On the horizontal axis is the cumulative distribution of “ability”
- On the vertical axis the difference between two cumulative curves:
- LOI: the cumulative value of the ICC under the assumption of statistical independence between the response and ability
- ACC: the cumulative value of ICC

Properties of LMA and NLMA

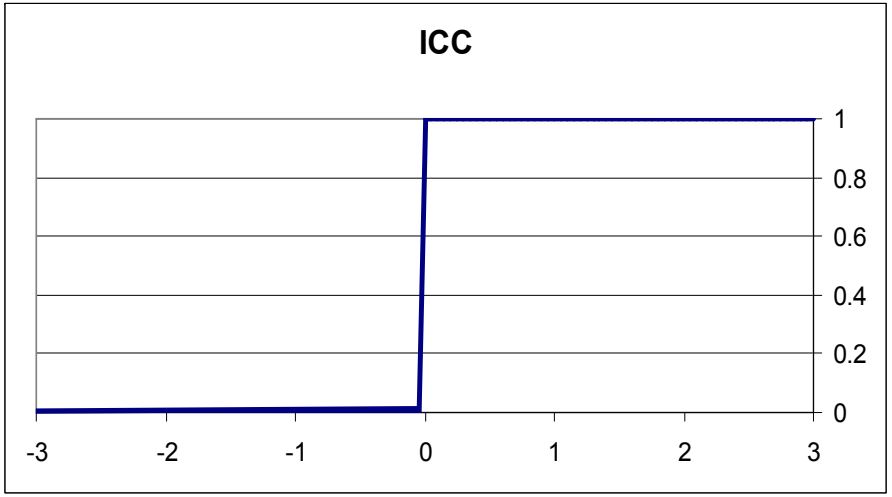
- **The curves offer a non-parametric substitute for the IRT model.**
- The curves start at (0,0) and end up at (1,0). They can take any shape depending on properties of $P(S|\theta)$.
- One can identify the quality of discriminatory power of the question by looking for peaks in the curves.

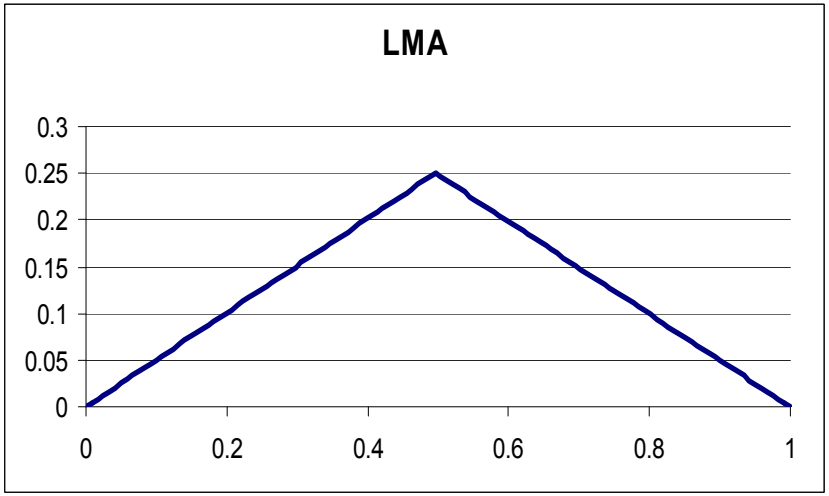
Properties of LMA

- If the curve is increasing (decreasing) then the probability of success is below (above) average.
- If the curve is concave (convex) then the ICC is (decreasing) increasing with ability
- The area enclosed between the curve and the horizontal axis is equal to
- COV [success, F(ability)]

The ICC and LMA curve

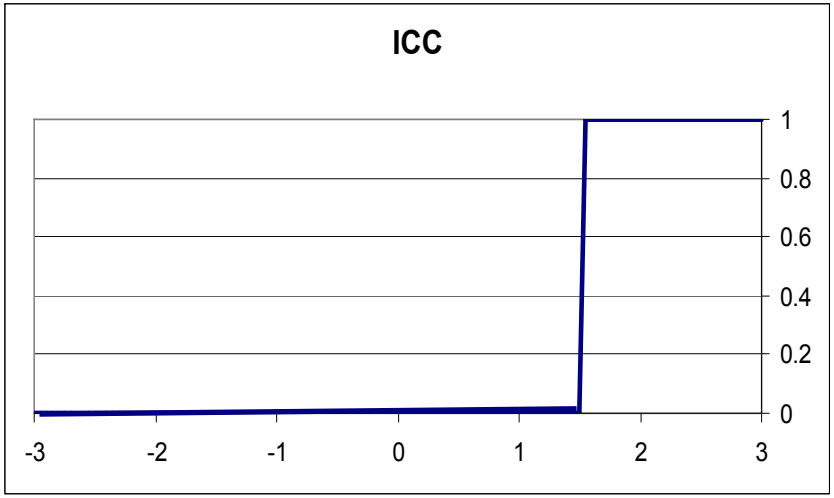
The Ideal Median-discrimination ICC

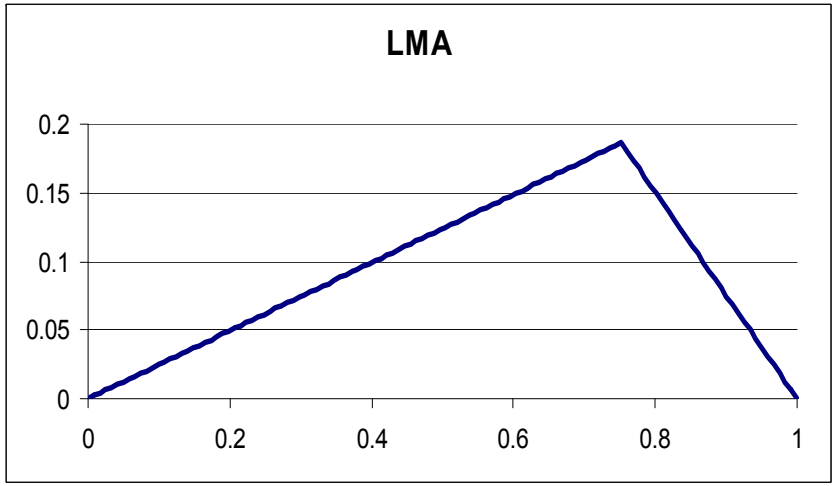




The ICC and LMA curve (*Cont.*)

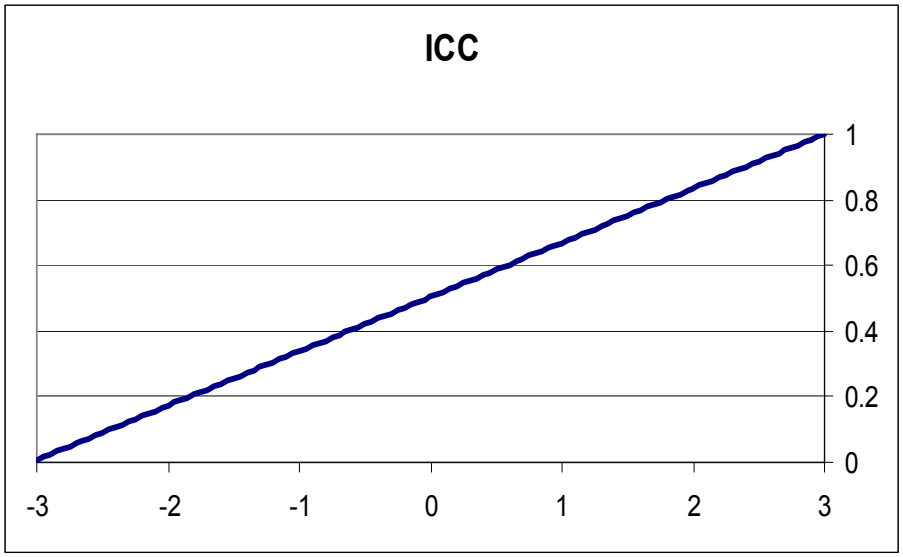
The Ideal ICC at the 0.8 discrimination

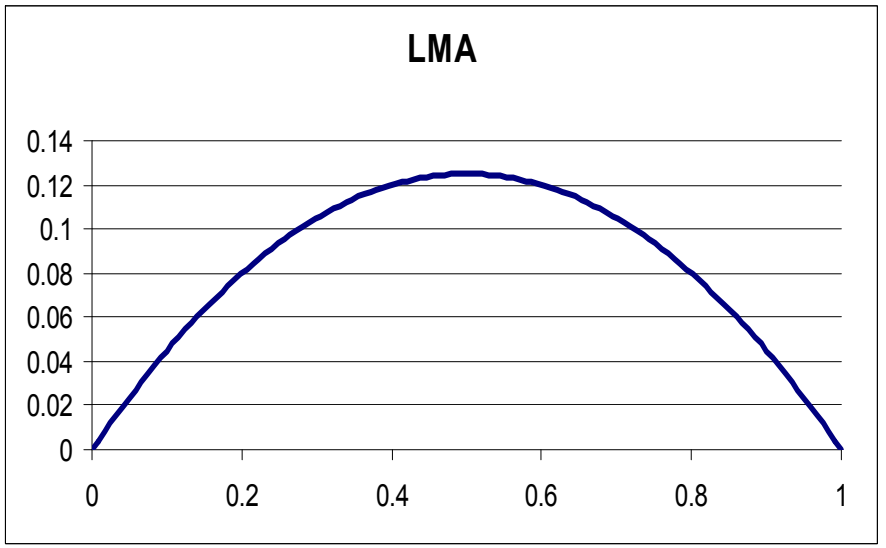




The ICC and LMA curve (*Cont.*)

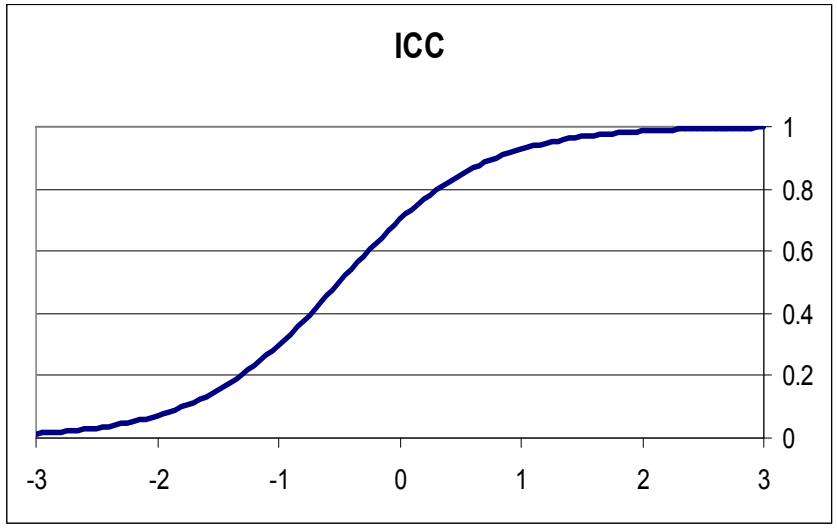
A Linear ICC

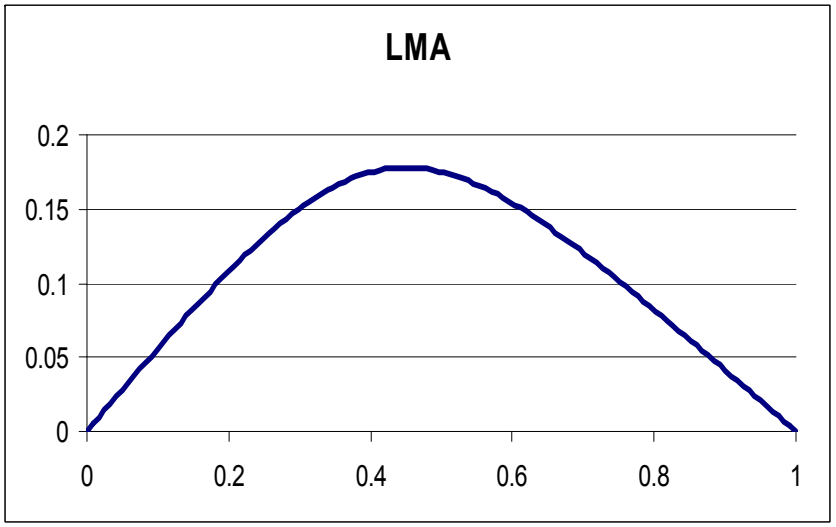




The ICC and LMA curve (*Cont.*)

A typical ICC





OLS and Gini regression coefficients

- The simple OLS regression coefficient

$$\beta^{OLS} = \frac{\mathbf{cov}(\mathbf{P}, \theta)}{\mathbf{cov}(\theta, \theta)}$$

- The Gini simple regression coefficient

$$\beta^{G} = \frac{\mathbf{cov}(\mathbf{P}, \mathbf{F}(\theta))}{\mathbf{cov}(\theta, \mathbf{F}(\theta))}$$

- Pearson correlation

$$\rho = \frac{\mathbf{cov}(\mathbf{P}, \theta)}{\sqrt{\mathbf{cov}(\mathbf{P}, \mathbf{P})\, \mathbf{cov}(\theta, \theta)}}$$

- Gini correlation coefficient

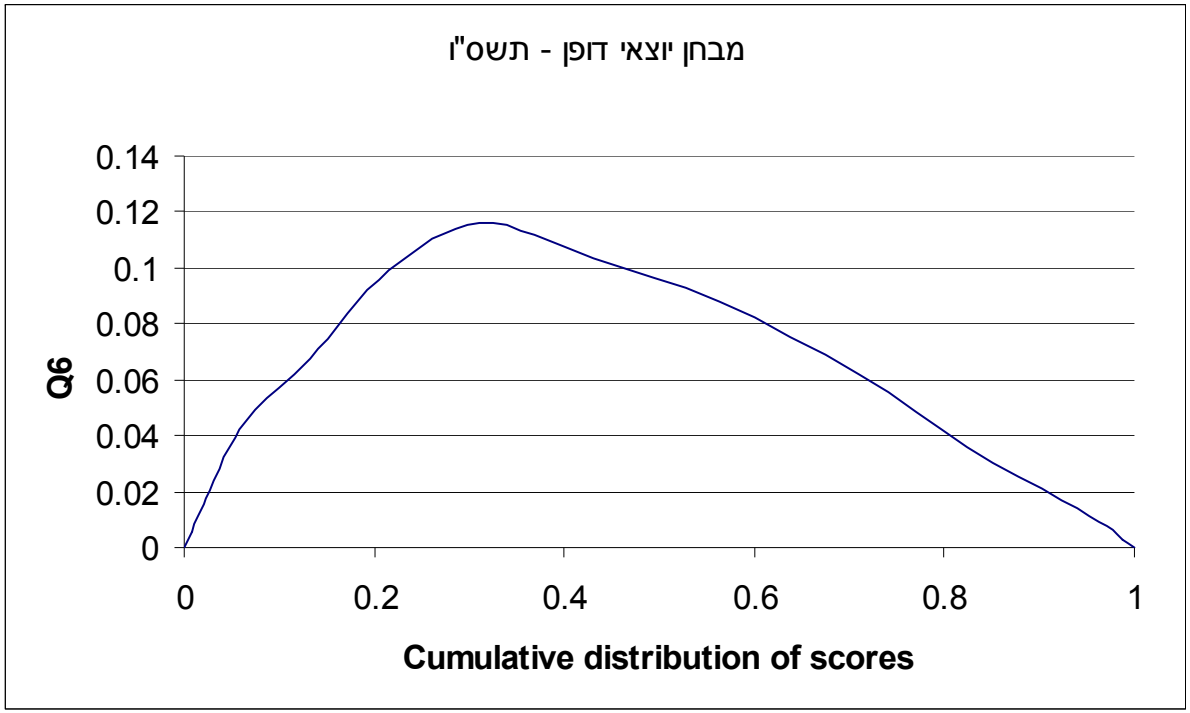
$$\Gamma = \frac{\mathbf{cov}(\mathbf{P}, \mathbf{F}(\theta))}{\mathbf{cov}(\mathbf{P}, \mathbf{F}(\mathbf{P}))}$$

Properties of LMA Curve (*Cont.*)

- If the LMA curve is above (below) the horizontal axis, and it does not intersect the horizontal axis then the correlation between answering the question and success in the exam is positive (negative), when measured along the full range of abilities.
- If the LMA curve intersects the horizontal axis then there are sections of ability with negative and positive correlation with the probability of a correct response to the question.

Illustrations

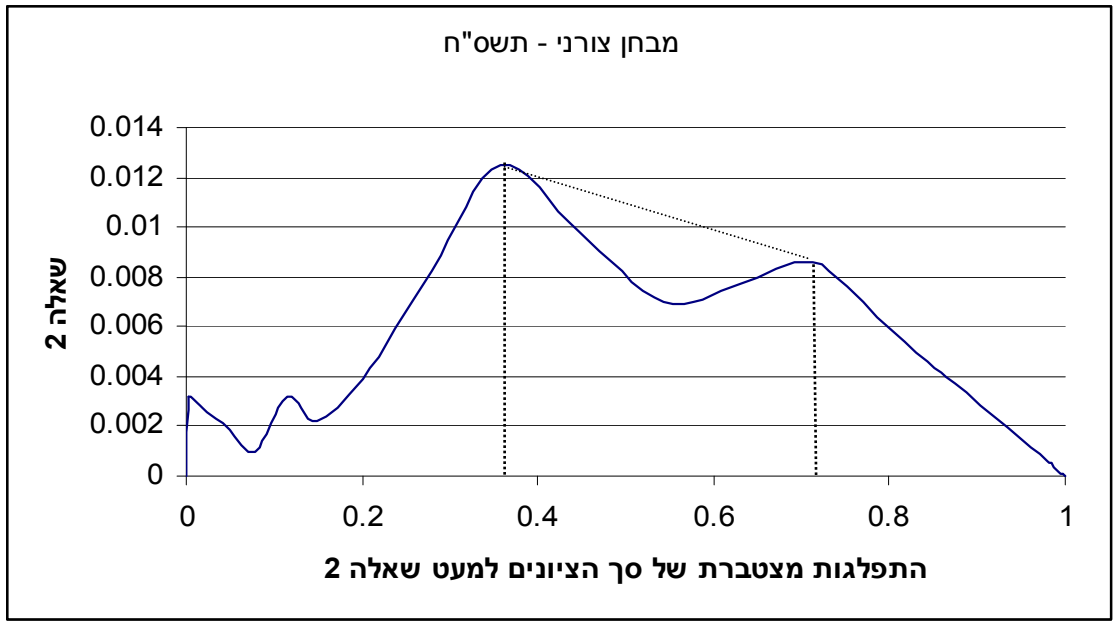
"Good" question



$\beta_{OLS}=0.094$, $\sigma_{OLS}=0.007$, $\beta_G= 0.093$, $\sigma_G=0.007$

Illustration (*Cont.*)

An inappropriate question?



Full regression: β_{OLS}=0.006, σ_{OLS}=0.002, β_G= 0.005, σ_G=0.002

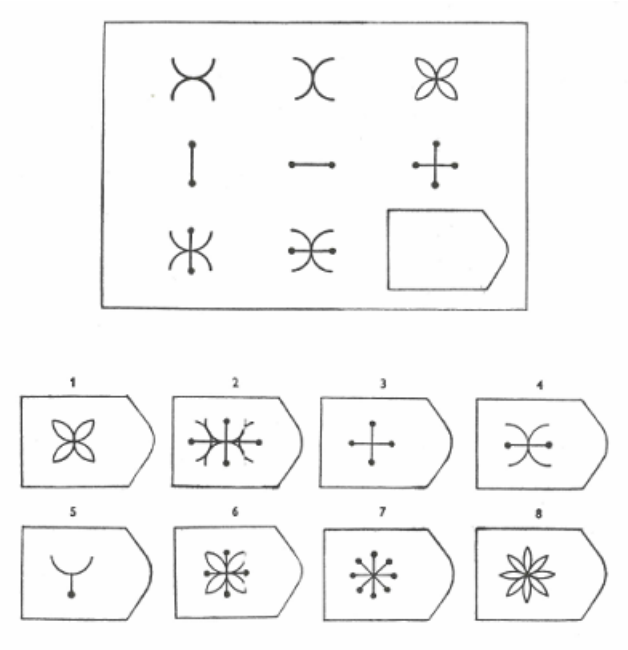
First section: β_{OLS}=0.010, σ_{OLS}=0.008, β_G= -0.005, σ_G=0.010

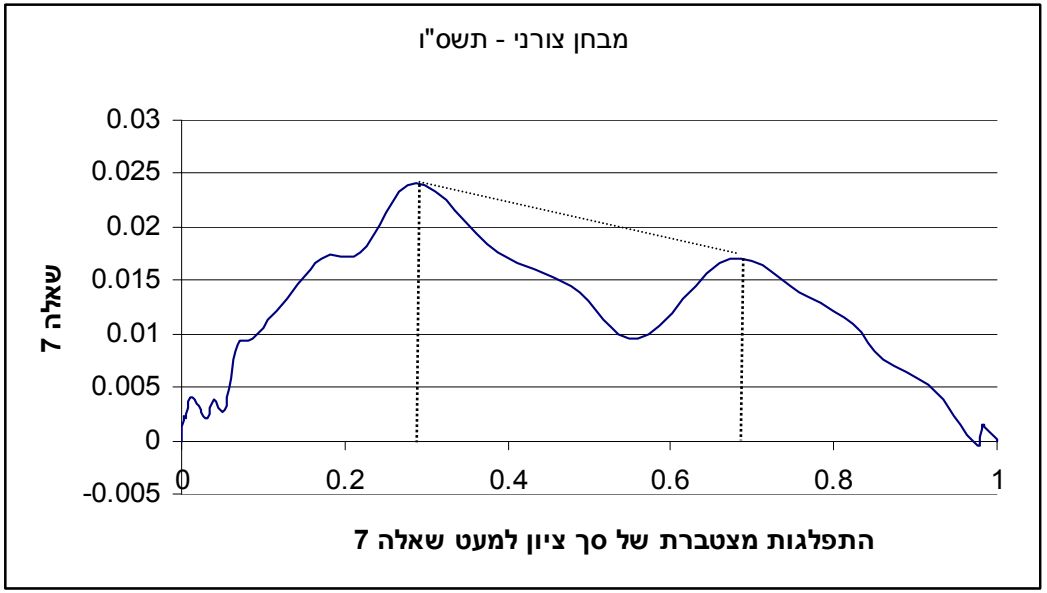
Without 2 obs.: β_{OLS}=-0.012, σ_{OLS}=0.010, β_G= -0.013, σ_G=0.009

Middle section: β_{OLS}=-0.016, σ_{OLS}=0.012, β_G= -0.016, σ_G=0.012

Illustrations (*Cont.*)

An "inappropriate" question?



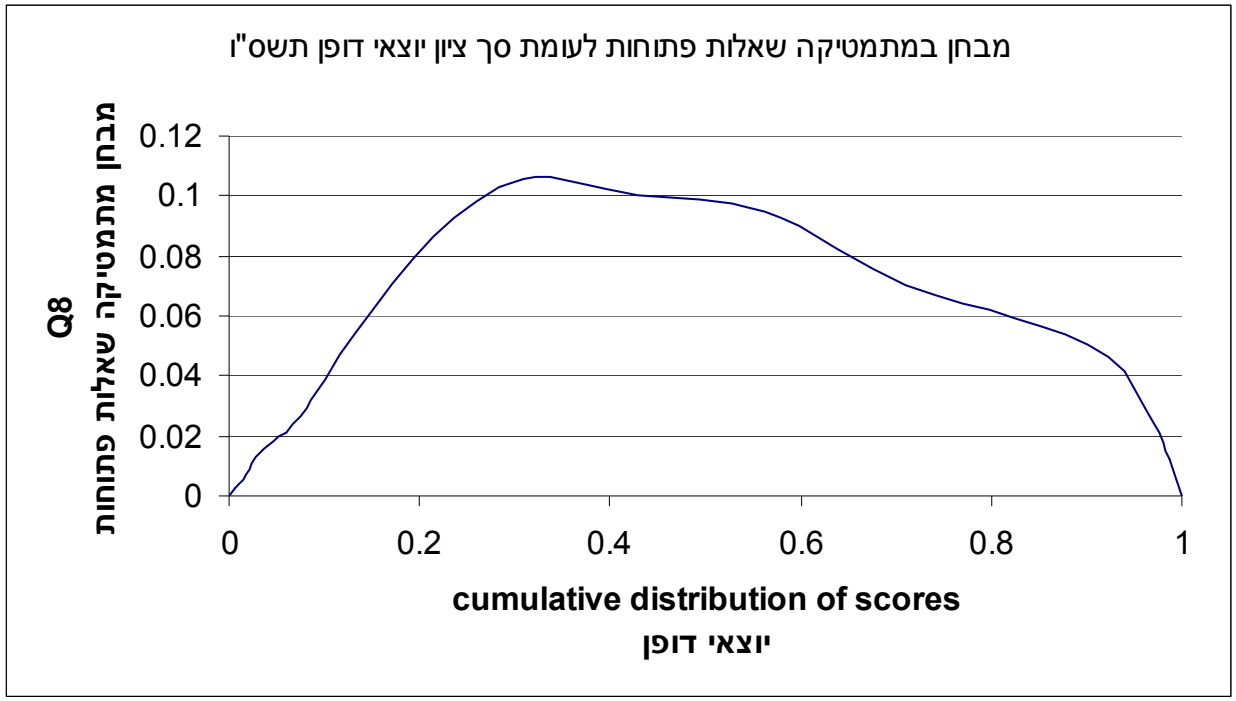


Full regression: β_{OLS}=0.011, σ_{OLS}=0.003, β_G= 0.010, σ_G=0.003

Middle section: β_{OLS}=-0.039, σ_{OLS}=0.018, β_G= -0.039, σ_G=0.020

Illustration – Contradictory check

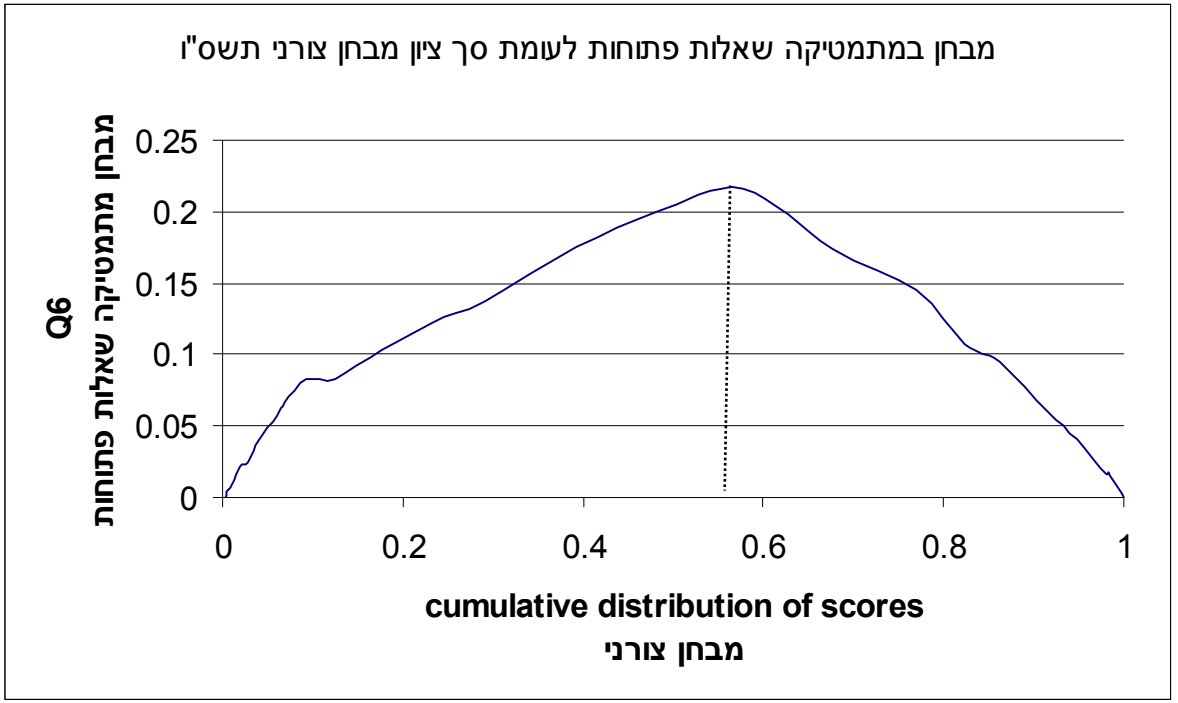
A redundant question?



$\beta_{OLS}=0.086$, $\sigma_{OLS}=0.017$, $\beta_G=0.087$, $\sigma_G=0.016$

Illustration – Contradictory check (*Cont.*)

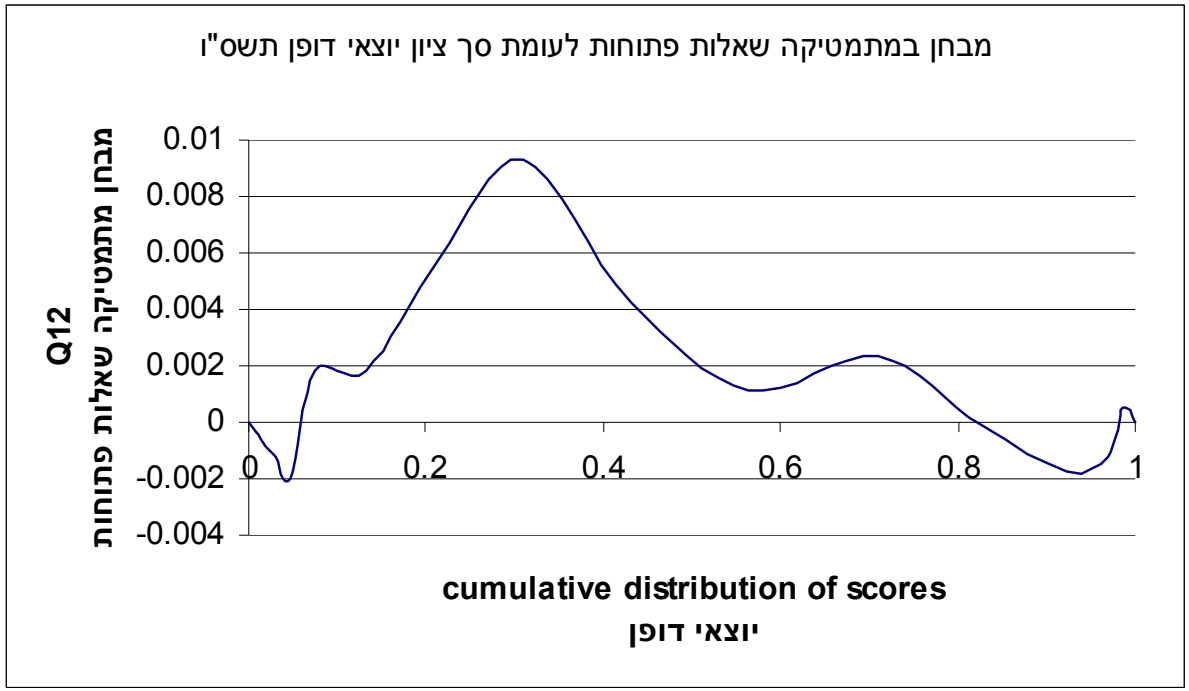
A redundant question?



$\beta_{OLS}=0.091$, $\sigma_{OLS}=0.013$, $\beta_G=0.100$, $\sigma_G=0.013$

Illustration – Contradictory check (*Cont.*)

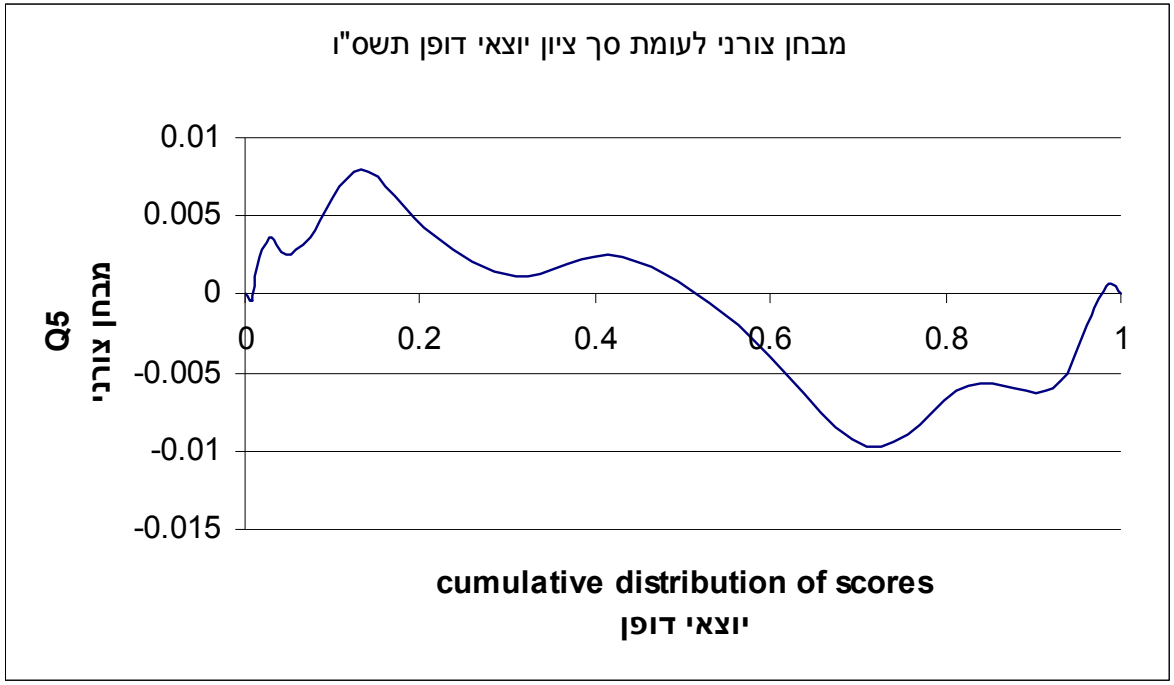
Different ability?



$\beta_{OLS}=0.003$, $\sigma_{OLS}=0.002$, $\beta_G=0.002$, $\sigma_G=0.002$

Illustration – Contradictory check (*Cont.*)

Different ability?



$\beta_{OLS}=0.000$, $\sigma_{OLS}=0.004$, $\beta_G= -0.001$, $\sigma_G=0.005$

Conclusions

- The LMA describes the monotonicity, the location and accuracy of the discriminative power of the question
- Two types of uses depending on whether the question belongs to the exam
- The software can be provided

LMA Curve

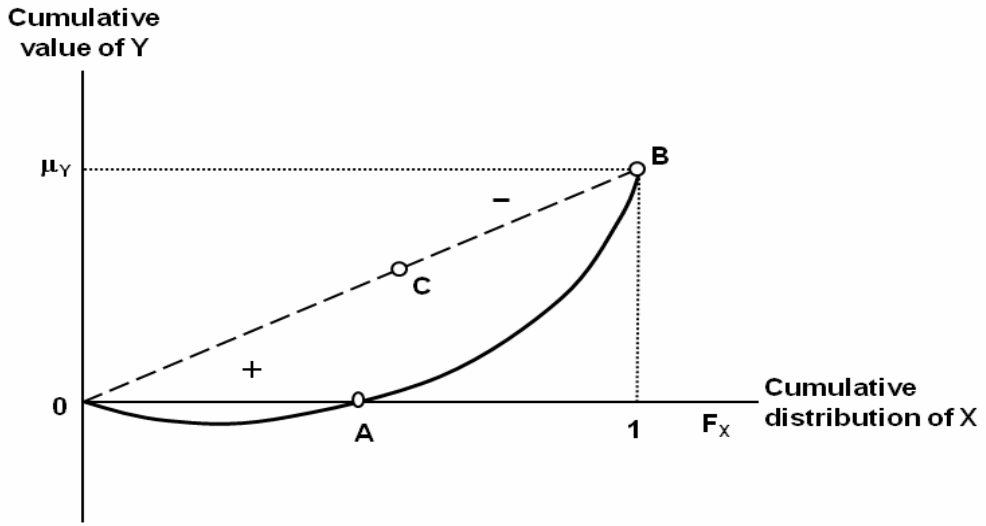
- LMA is Line of independence Minus Absolute concentration curve:
- Let μ =E{S} be the expected value of success in answering the question among the examinees,
- Let $F_\theta = F(\theta)$ be the cumulative distribution of ability.
- **L**ine **O**f **I**ndependence is: $LOI(F_\theta^*) = \mu F_\theta^*$
- **A**bsolute **C**oncentration **C**urve is:

$$ACC(F_\theta^*) = \int_0^{F_\theta^*} P(S|F_\theta)dF_\theta$$

LMA Curve (*Cont.*)

- The LMA curve is defined as the vertical difference between LOI and ACC curves:

$$\mathbf{LMA}(\mathbf{F}_{\theta}^{*}) = \mathbf{LOI}(\mathbf{F}_{\theta}^{*}) - \mathbf{ACC}(\mathbf{F}_{\theta}^{*})$$

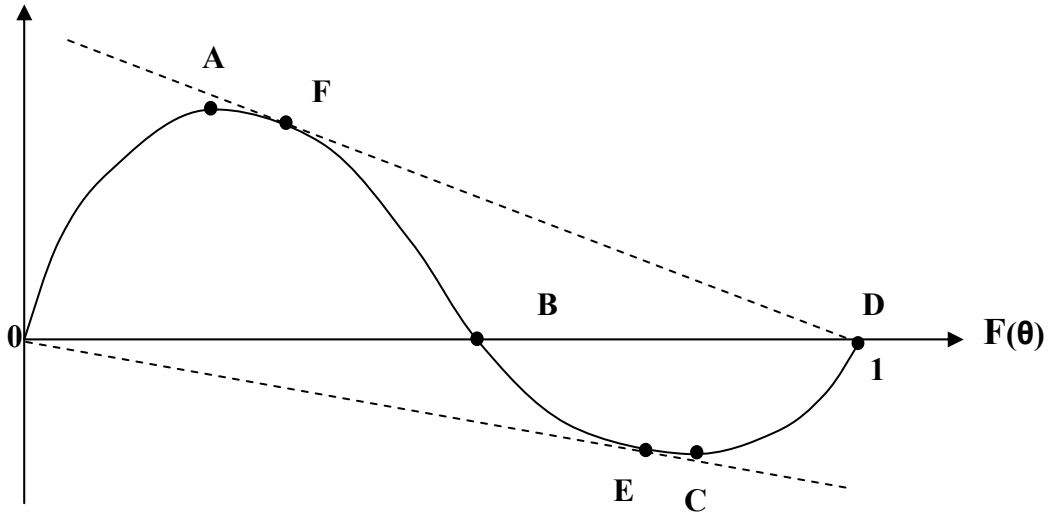


- The area enclosed between the LMA curve and the horizontal axis is equal to cov (P, F(θ)).

NLMA Curve

- The NLMA curve is a Normalized LMA curve, accepted by dividing the vertical axis by cov(θ,F(θ)).

$$NLMA(\frac{\mu - E\{P(\theta) \mid \theta_F\}}{Cov(\theta, F(\theta))})$$



- The area enclosed between the curve and the horizontal axis is equal to the Gini regression coefficient.

Properties of LMA Curve

- If the curve increases (decreases, horizontal) then the probability of success in answering the question is above (below, equal) average success in answering the question.
- If the LMA curve is concave (linear, convex) then the probability of answering the question is increasing (constant, decreasing) locally.

Curves and regression coefficients

- If the curve is always above (below) the horizontal axis, then the sign of Gini and OLS regression coefficients of P on θ will be positive (negative) for all monotonic non-decreasing transformations of P or θ.
- If the curve intersects the horizontal axis then there exists a monotonic non-decreasing transformation of θ that can change the sign of the Gini and the OLS regression coefficients.
- If in a given section the curve is convex (concave) then the sign of Gini regression coefficient in that section are negative (positive).